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Electrodynamical Origins of Einstein's Theory of General Relativity

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The failure of the Newtonian theory of gravitation to satisfactorily account for the motion of Mercury's perihelion cannot be held to have justified the development of general relativity. This paper shows how the origins of general relativity were firmly embedded in contemporary attempts to introduce the new mechanics of special relativity into gravitational theory. These new theories of gravitation took as their basis the electrodynamical equations as formulated by Minkowski and attempted to represent the gravitational potential first by a vector and then by a scalar (in the four-dimensional sense). That Einstein chose the symmetric fundamental tensor g_{ij} as his gravitational potential is seen to have been both a natural and necessary development. With this viewpoint the full theory of general relativity can be seen to be remarkably similar to those theories of gravitation that preceded it. The paper also contains a previously unpublished letter written by Einstein to H. A. Lorentz.

1. INTRODUCTION

Let us consider two reference systems Σ_1 and Σ_2. Σ_1 is accelerating in the direction of its X-axis, and let γ be the magnitude (timelike, constant) of this acceleration. Σ_2 is at rest; but finds itself, however, in a homogenous gravitational field, which imparts to all matter the acceleration $-\gamma$ in the direction of the X-axis.

As far as we know, the physical laws in reference to Σ_1 do not differ from those in reference to Σ_2; it is because of this that all bodies in the gravitational field are accelerating equally. We have therefore, in the case of the present position of our knowledge, no motive to the assumption that the systems Σ_1 and Σ_2 differ in any respect from one another and therefore will, in the

> following discussion, assume the full physical equivalence of the gravitational field and the corresponding acceleration of the reference system. (Einstein, 1907).

This was the first enunciation of the principle of equivalence and immediately brought relativity theory and gravitation together under the same umbrella, since according to the principle, a homogeneous gravitational field can be replaced by an accelerating reference system. With hindsight it is known that the principle of equivalence, as stated above, is no longer a prerequisite to the establishment of the theory; indeed Synge goes so far as to say, "I suggest that it be now buried with appropriate honours." From an historical point of view, however, this is not the case, since it was via the principle that Einstein was led to conject that space-time was not Euclidean, but Riemannian. Put another way this means that in order to develop the theory of relativity, the Lorentz group is no longer the universal symmetry group and instead we have to consider the group of all continuous, differentiable coordinate transformations with nonvanishing Jacobian; this is the mathematical representation of the principle of equivalence.

From this it seems that in extending the theory of special relativity to include accelerated frames of reference and hence via the principle of equivalence, gravitation, ds^2 no longer has its Minkowski form but is a general quadratic function of the coordinate differentials, and generally,

$$ds^2 = g_{ij} dx^i dx^j$$

where $i, j, = 1, 2, 3, 4$ for space-time.

Thus the principle of equivalence leads us directly to the principle of general covariance, which itself is just the mathematical representation of the former principle, and therefore directly to the hypothesis that the physical four-dimensional continuum has a Riemann metric. This was the heuristic value of the principle of equivalence; it enabled Einstein to form a workable basis for the general theory of relativity and to establish in his mind exactly what was required to complete the theory. Basically two things remained, at this stage, to be tackled:

1. The equations of special relativity must be transformed into equations which are covariant with respect to nonlinear substitutions.

2. The law of the gravitational field must be found, i.e., second-order differential invariants of g must be found. The ensuing chapters will show that the first problem concerned itself with the tensor calculus of Ricci and Levi-Cività, and, as far as electromagnetic equations were concerned, had already been solved by F. Kottler, (Kottler, 1912) and that Einstein fol-

lowed the example set by Kottler. The second problem had been considered by Riemann himself in connection with the tensor that now bears his name, and indeed Einstein had in fact successfully obtained the correct field equations in his paper with Grossmann (Einstein and Grossmann, 1913), but neither of them recognized the equations as physically applicable. Indeed, Einstein even convinced himself that because of arguments concerning causality, it was impossible to form a law of gravitation which was invariant for all possible transformations of the coordinates.

No researcher works in isolation of those who are working in the same field, and this is especially true of Einstein during this period. Before he ventured along the little-traveled road of invariant theory, Einstein had attempted to form a theory of gravitation along more conventional lines. The remainder of this article will show that a tensor theory of gravitation was as logical a step in the development of theories of gravitation as was the advent of special relativity in the development of the theories of electrodynamics. This pressure, together with the implications of the principle of equivalence, led Einstein to general relativity.

2. NEWTONIAN GRAVITATION

There exists a vast literature on Newtonian gravitational theories and here is not the place to repeat them. However, something must be said as to why these theories were unsatisfactory, and I will do this by means of a little-known example which illustrates the depths to which past researchers were willing to go in order to salvage the old theory, and in fact, the ingenuity of these attempts. U. J. J. Leverrier carried out many calculations on the interactions of the planets of the solar system, analyzing the perturbation effects that each has on the other. It was he who showed the existence of probably the most famous "anomaly" of planetary motion, i.e., the anomalous motion of Mercury's perihelion. Due to various known effects the orbit of Mercury advances 5,599.7 seconds of an arc per century, and through his calculations, Leverrier found that some 41.25 seconds of an arc per century could not be accounted for by the above known phenomena. Rather than reject the theory on which these calculations were based (the theory had had some notable successes such as the discovery of Neptune), Leverrier and later J. Bauschinger hypothesized the existence of other undiscovered planets or some sort of planetary matter. We will call these hypotheses mass hypotheses, of which there were many, and investigate one of them, that put forward by H. V. Seeliger (1901; 1906a,b).

As his seat of matter Seeliger chose a phenomenon called the zodiacal light, the discovery of which is attributed to J. D. Cassini.[1] The masses which reflect the zodiacal light were supposed to have the form of a flat disk surrounding the Sun and extending nearly in the direction of the orbital planes of the planets, reaching outside the orbit of the earth, and a maximum density equivalent to about 1/27th of a cubic meter of water per cubic kilometer of space. Seeliger imagined the disk shape to be made up of a number of extremely flattened ellipsoids of rotation superposed together so that the density varies after a certain law from the center outwards, as required. At first he introduced five such ellipsoids but later found that two would suffice. In order to reach agreement with Newcomb's residuals, he also added the hypothesis that a rotation of the empirical system of coordinates with respect to the "inertial system" must be taken into account.

The theory can thus be reduced to three hypotheses:

1. Attraction caused by an ellipsoid existing entirely within the orbit of Mercury. Conveniently, the proximity of this ellipsoid to the sun makes it invisible.

2. Attraction caused by an ellipsoid which includes the earth's orbit and gives rise to the zodiacal light.

3. A rotation of the empirical coordinate system of astronomy, i.e., the "fixed stars."

The ingenuity of Seeliger's theory can be judged from the table of residuals comparing his calculations and those of Newcomb (Table I). In this table e is the eccentricity, π the perihelion, Ω the longitude of node, and i the angle of inclination.

Criticism of Seeliger's theory was not immediately forthcoming, but in 1913 J. Woltjer and W. de Sitter examined Seeliger's results in minute detail in order to find any serious discrepancy with observation. It is a credit to the merit of the theory that criticism had to take such a form. In order to analyze the results de Sitter produced a table of his own which presented Newcomb's residuals, Seeliger's residuals, and residuals that would be obtained if hypotheses 2 or 3 were denied (Table II). Much of the table consists of data already given in Table I, so only additional information is listed. The rows labeled A are the residuals obtained by rejecting hypothesis 3. The rows labeled B are the residuals obtained by rejecting hypothesis 2.

By a straightforward analysis of these residuals one can see, for instance, that the residuals obtained by assuming hypothesis B are just as good as those of Seeliger's theory, indicating that the second ellipsoid is superfluous, and that because of the residual $+1.18''$, which is entirely

[1] It is possible that it was known before this time; perhaps *Romeo and Juliet*, Act III, scene V, gives us a clue: "Yon light is not daylight, as I know it: It is some meteor that the sun exhales."

TABLE I

Quantity	Planet	Newcomb residual	Seeliger residual	Mean error
	Mercury	+8.48″	−0.01″	±0.43″
	Venus	−0.05″	−0.10″	±0.25″
$e\,\Delta\pi$	Earth	+0.10″	+0.03″	±0.13″
	Mars	+0.75″	+0.16″	±0.35″
	Mercury	+0.61″	−0.04″	±0.52″
$\sin i\,\Delta\Omega$	Venus	+0.60″	−0.02″	±0.17″
	Mars	+0.03″	−0.20″	±0.22″
	Mercury	+0.38″	−0.14″	±0.80″
Δi[a]	Venus	+0.38″	+0.21″	±0.33″
[3]	Earth	−0.22″	+0.28″	±0.27″
	Mars	−0.01″	+0.01″	±0.20″
	Mercury	−0.88″	—	±1.50″
Δe	Venus	+0.21″	—	±0.31″
	Earth	+0.02″	—	±0.10″
	Mars	+0.29″	—	±0.27″

[a] This value was not actually calculated by Seeliger but by J. Woltjer (Woltjer, 1914; de Sitter, 1914) using the same method.

inadmissible, the rotation of the empirical coordinate system seems to be the most important feature of the theory. The ad hoc nature of the theory is thus exposed, and like all of the theories that preceded it, Seeliger's theory suggests that it was the underlying Newtonian mechanics of the system that must be at fault.

It is of interest, here, to note that if one uses an approximation method to solve Einstein's gravitational equations it is found that the results indicate that there is indeed a rotation of the empirical system of coordinates, and when applied to the motion of the moon, there is an advance in the perigee

TABLE 2

	Mercury	Venus	Earth	Mars
$e\,\Delta\pi$ A	0.00″	−0.05″	+0.18″	+0.52″
B	−0.02″	−0.12″	−0.04″	0.00″
$\sin i\,\Delta\Omega$ A	+0.55″	+0.01″	—	−.11″
B	−0.31″	+0.05″	—	−0.24″
Δi A	−0.12″	+0.17″	+1.18″	+0.05″
B	−0.15″	+0.23″	−0.17″	−0.01″

and node of some 1.94″ per century. According to Seeliger's theory we have

$$\frac{d\tilde{\omega}}{dt} = +2.11'', \qquad \frac{d\Omega}{dt} = -2.50''$$

while

$$\text{(A)} \qquad \frac{d\tilde{\omega}}{dt} = +2.04'', \qquad \frac{d\Omega}{dt} = -3.30''$$

$$\text{(B)} \qquad \frac{d\tilde{\omega}}{dt} = +2.10'', \qquad \frac{d\Omega}{dt} = -2.06''$$

These results seem to indicate that it is indeed the rotation of the chosen inertial frame with respect to the general inertial frame of the sidereal system (empirical system of coordinates) that is the cause of the discrepancy between theory and observation. One might be tempted to suggest therefore that Newtonian theory may be retained if a suitable rotation of coordinates is chosen, but it must be emphasized that the rotation, as the relativistic analysis shows, is essentially non-Newtonian, and arguments as to the nature of the rotation are between relativists and nonrelativists, both of whom, however, are non-Newtonian.

The above example was chosen so as to summarize, fairly succinctly, the reasons why Newton's laws of gravitation had to be replaced and why these new theories applied themselves to the very underlying mechanics and the underlying philosophy of the problem rather than repeating the errors of former years, in only producing superficially different theories; the advent of special relativity was the catalyst for this change in emphasis.

3. SCALAR AND VECTOR THEORIES OF GRAVITATION

The influence of the theory of relativity was, as was to be expected, first felt in the field of electrodynamics and given additional impetus when Minkowski introduced his concept of space-time and proceeded to apply the results of his ideas to the electrodynamics of moving bodies. The years between 1905 and 1912 saw the results of Einstein and Minkowski being applied to electrodynamics by many researchers: Laue, Abraham, Lorentz, for example, in the context of establishing a cogent and consistent picture of the electromagnetic field. We may summarize most of the results in a covariant notation as follows.

Lorentz himself had shown that the force per unit volume (force density) $\mathbf{f}$ can be represented as the resultant of the surface forces, which are produced by the Maxwell stresses and the negative time derivative of the momentum density of the ether. The stress tensor is defined by (Lorentz, 1904)

$$T_{ik} = \left(E_i E_k - \tfrac{1}{2}\mathbf{E}^2\delta_{ik}\right) + \left(H_i H_k - \tfrac{1}{2}\mathbf{H}^2\delta_{ik}\right), \qquad i, k = 1,2,3$$

and the electromagnetic momentum by

$$\mathbf{g} = \frac{1}{c^2}\mathbf{S}, \qquad \mathbf{S} = c(\mathbf{E} \wedge \mathbf{H})$$

Then

$$\mathbf{f} = \operatorname{div} T - \dot{\mathbf{g}}$$

where T is the stress tensor whose components are T_{ik}. This vector equation together with the energy equation

$$\frac{\partial w}{\partial t} + \operatorname{div}\mathbf{S} = -\mathbf{f}\cdot\mathbf{m}, \qquad w = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2)$$

can be combined to form a four-vector equation

$$f_i = -\frac{\partial S_{ik}}{\partial x^k}, \qquad i, k = 1,2,3,4 \tag{1}$$

where S_{ik} is a symmetrical tensor called the energy momentum tensor whose components are such that

$$S_{ik} = -T_{ik} \qquad \text{for } i, k = 1,2,3$$

$$(S_{14}S_{24}S_{34}) = (S_{41}S_{42}S_{43}) = \frac{i}{c}\mathbf{S} = ic\mathbf{g}$$

and

$$S_{44} = -W$$

The importance of equation (1) is that for $i = 1,2,3$ it represents the law of conservation of momentum, while for $i = 4$ it represents energy conservation.

The above results, which are initially derived in the context of electrodynamics, were used to form the basis of a general dynamics, which in turn, as we shall see, was used to form the basis of several theories of gravitation. Basically the transition from electrodynamics to gravitational theory was performed by means of the principle of the inertia of energy, by which to each energy E there corresponds a mass $M = E/c^2$. What the above equations tell us, however, is that E need not be the total energy of a system, as was first hypothesized, but that a mass (or momentum) may be associated with each individual energy (or energy current). This result follows from the symmetry of the energy-momentum tensor, for

$$S_{ik} = S_{ki}, \qquad i, k = 1, 2, 3$$

and hence

$$\mathbf{g} = \mathbf{S}/c^2$$

Lorentz, O. F. Mossotti, and F. Zollner between them considered the gravitational field and its equations as analogous to the electromagnetic field and its equations such that the expression for the force of gravity referred to unit matter is given by

$$\mathbf{f}^g = \mathbf{E}^g + \frac{1}{c}(\mathbf{M} \wedge \mathbf{H}^g)$$

and the differential equations of the static gravitational field are

$$\operatorname{div} \mathbf{E}^g = -\rho_m$$

and

$$\mathbf{E}^g = -\operatorname{grad} \phi$$

where ϕ is the gravitational potential, ρ_m the mass density, and units have been chosen appropriately. The equation for the energy density in the above case is analogously

$$W^g = -\tfrac{1}{2}(\mathbf{E}^g)^2$$

and for the nonstatic case

$$W^g = -\tfrac{1}{2}\left[(\mathbf{E}^g)^2 + (\mathbf{H}^g)^2\right]$$

The defects of this form of the theory are fairly obvious. First, the force of gravity is hypothesized to be velocity dependent; this has not been experimentally verified. Second, since the expression for the energy density is negative this means that in any region of space, its energy is decreased if a gravitational field is introduced. It should be noted at this point that the theory as developed by Poincaré in the last section of his Rendiconti paper also suffered from these defects.

Now as suggested above this theory of gravitation can take as its basis those equations already developed; thus the conservation laws of the gravitational field are given by

$$f_i^g = -\frac{\partial S_{ik}^g}{\partial x^k} \tag{2}$$

where the g is not a tensor index but a superscript denoting the gravitational case. As in electromagnetism, the quantity $\mathbf{E}^g$ can be derived from the vector and scalar potential resulting in the fact that S_{ik}^g is now to be considered a function of the four-vector potential ϕ_i through the equation

$$F_{ik} = \frac{\partial \phi_k}{\partial x^i} - \frac{\partial \phi_i}{\partial x^k}$$

For this reason the above-outlined theory and others similar to it were known as vector theories.

An obvious alternative to the vector theories of gravitation is to form a theory whose basis is the assumption that the gravitational potential is derived from a scalar. In the years preceding Einstein's theory, these scalar theories, as they were called, dominated the gravitational scene, as can be seen simply by noting the list of prominent scientists who made contributions to the subject besides Einstein himself: Abraham, Nordström, and Mie, to name but a few. We shall analyze the most important of these theories, their advantages and disadvantages, to see how their failure and the failure of the vector theories made a tensor theory of gravitation (in the sense that the gravitational potential is a tensor) the logical alternative.

The difficulties of the vector theories of gravitation, as far as the energy density were concerned, were largely overcome by assuming that the gravitational potential depended on a scalar rather than a vector. Abraham, taking his cue from a paper by V. Volterra, set up a scalar gravitational theory in the following way (Abraham, 1912a).

The gravitational force per unit volume has the following components:

$$f_1^g = -\rho_{m_0}\frac{\partial \phi}{\partial x^1}, \qquad f_2^g = -\rho_{m_0}\frac{\partial \phi}{\partial x^2}$$

$$f_3^g = -\rho_{m_0}\frac{\partial \phi}{\partial x^3}, \qquad f_4^g = -\rho_{m_0}\frac{\partial \phi}{\partial x^4}$$

where ρ_{m_0} is, as yet, an undetermined factor. Now through direct comparison with electrodynamics, in which

$$f_4 = \frac{i\rho}{c}(\mathbf{M} \wedge \mathbf{E})$$

where **M** is the velocity of the matter with respect to some frame, we have

$$\frac{c}{i} f_4^g = ic\rho_{m_0} \frac{\partial \phi}{\partial x^4} = \rho_{m_0} \frac{\partial \phi}{\partial t}$$

which describes the rate at which the gravitational forces are doing work. Thus we have

$$f_i^g = -\rho_{m_0} \frac{\partial \phi}{\partial x^i}$$

and

$$f_i^g = -\frac{\partial S_{ik}^{g(\phi)}}{\partial x^k}$$

where this time the superscript $g(\phi)$ denotes not only the case of gravitation but also the fact that $S_{ik}^{g(\phi)}$ depends on the scalar ϕ. In detail we have then

$$S_{11}^{g(\phi)} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x^1}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x^2}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x^3}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x^4}\right)^2$$

$$S_{22}^{g(\phi)} = -\frac{1}{2}\left(\frac{\partial \phi}{\partial x^1}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x^2}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x^3}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x^4}\right)^2$$

$$S_{33}^{g(\phi)} = -\frac{1}{2}\left(\frac{\partial \phi}{\partial x^1}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x^2}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x^3}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x^4}\right)^2$$

$$S_{44}^{g(\phi)} = -\frac{1}{2}\left(\frac{\partial \phi}{\partial x^1}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x^2}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x^3}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x^4}\right)^2$$

$$S_{12}^{g(\phi)} = S_{21}^{g(\phi)} = \frac{\partial \phi}{\partial x^1} \frac{\partial \phi}{\partial x^2}$$

$$S_{23}^{g(\phi)} = S_{32}^{g(\phi)} = \frac{\partial \phi}{\partial x^2} \frac{\partial \phi}{\partial x^3}$$

$$S_{31}^{g(\phi)} = S_{13}^{g(\phi)} = \frac{\partial \phi}{\partial x^1} \frac{\partial \phi}{\partial x^3}$$

$$S_{14}^{g(\phi)} = S_{41}^{g(\phi)} = \frac{\partial \phi}{\partial x^1} \frac{\partial \phi}{\partial x^4}$$

$$S_{24}^{g(\phi)} = S_{42}^{g(\phi)} = \frac{\partial \phi}{\partial x^2} \frac{\partial \phi}{\partial x^4}$$

$$S_{34}^{g(\phi)} = S_{43}^{g(\phi)} = \frac{\partial \phi}{\partial x^3} \frac{\partial \phi}{\partial x^4}$$

As before, analogously

$$c^2\mathbf{g}^g = \mathbf{S}^g = -\frac{\partial\phi}{\partial t}\operatorname{grad}\phi$$

and

$$\begin{aligned} W^g &= -S_{44}^{g(\phi)} \\ &= \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x^1}\right)^2 + \left(\frac{\partial\phi}{\partial x^2}\right)^2 + \left(\frac{\partial\phi}{\partial x^3}\right)^2 - \left(\frac{\partial\phi}{\partial x^4}\right)^2\right\} \\ &= \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x^1}\right)^2 + \left(\frac{\partial\phi}{\partial x^2}\right)^2 + \left(\frac{\partial\phi}{\partial x^3}\right)^2 + \frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2\right\} \end{aligned}$$

represent the momentum density and the energy density, respectively, of the gravitational field.

It should be noted that as in electrodynamics these equations represent the energy and momentum conservation laws and that, significantly, W^g, the energy density, is positive as required. By extending the usual vector analysis to four dimensions we have

$$\frac{\partial S_{ik}^{g(\phi)}}{\partial x^k} = \Box\phi\frac{\partial\phi}{\partial x^i}$$

where

$$\Box\phi = \frac{\partial^2\phi}{\partial x^{i^2}}$$

but from equation (2) comes

$$f_i^g = -\Box\phi\frac{\partial\phi}{\partial x^i}$$

and therefore from

$$f_i^g = -\rho_{m_0}\frac{\partial\phi}{\partial x^i}$$

we have

$$\Box\phi = \rho_{m_0} \tag{3}$$

which in the context of this theory is taken as the generalization of Poisson's equation so that ρ_{m_0} can therefore be interpreted as the rest mass density in the four-dimensional sense. This equation is historically extremely important in that it marks the point where the theories of Abraham and Einstein crossed; the point in fact where Einstein, in his characteristically stubborn way, chose to go his own way in insisting that it was his principle of equivalence, which first indicated the need to extend the principle of special relativity, that must act as the guideline along which any gravitational theory must develop. This is the point where Einstein differs from all other researchers.

Both Einstein and Abraham had come to the conclusion (Einstein some four years earlier than Abraham), that the gravitational potential is a function of the velocity of light, but differed as to their conclusions as to the effect this hypothesis had on the nature of space and time. Abraham thought that the Lorentz transformations would still be valid, just as in special relativity (constant c), in infinitely small space-time regions where c may be considered as constant; consequently Einstein's principle of relativity would only be valid in such isolated systems. Einstein did not want to give up his principle, because to him the idea of limiting the principle to the case of uniform motion between systems elevated the concept of inertial systems to an unwarrantable position and further retained the absolute nature of acceleration. Within his own theory at this time such restraints on the validity of the principle led to contradictory results. If Abraham's argument was valid then in an infinitely small space-time region the Lorentz transformations become

$$dx' = \frac{dx - v\,dt}{(1 - v^2/c^2)^{1/2}}$$

$$dt' = \frac{(-v/c^2)\,dx + dt}{(1 - v^2/c^2)^{1/2}}$$

where $v = |\mathbf{m}|$. Written in a more workable form these become

$$dx' = \beta\,dx - v\beta\,dt$$

$$dt' = -\frac{v}{c^2}\beta\,dx - \beta\,dt$$

where β is the relativistic factor $1/(1 - v^2/c^2)^{1/2}$. Now dx' and dt' must be perfect differentials, hence

$$\frac{\partial\beta}{\partial t}=\frac{\partial(-v\beta)}{\partial x}$$

and

$$\frac{\partial[-(v/c^2)\beta]}{\partial t}=\frac{\partial\beta}{\partial x}$$

which when written in full become

$$\begin{aligned}\frac{\partial}{\partial t}\left\{\frac{1}{(1-v^2/c^2)^{-1/2}}\right\}&=\frac{\partial}{\partial x}\left\{\frac{-v}{(1-v^2/c^2)^{-1/2}}\right\}\\ \frac{\partial}{\partial t}\left\{\frac{-v/c^2}{(1-v^2/c^2)^{-1/2}}\right\}&=\frac{\partial}{\partial x}\left\{\frac{1}{(1-v^2/c^2)^{-1/2}}\right\}\end{aligned}\tag{4}$$

If it is now understood that the above equations are valid for an infinitely small region of space-time in which there exists a static gravitational field, then c is an arbitrarily given function of x but independent of t. If the dashed system is supposed to be in uniform motion, then v must be independent of x and necessarily independent of t. From this analysis therefore we conclude that the left-hand side of equations (4), and consequently the right-hand side, must vanish accordingly. This latter conclusion is, however, impossible, since in the case of arbitrary functions of x for given c both right-hand sides cannot be made to disappear by choosing v as a suitable function of x. Herein lies the notion that as soon as one gives up the universal constancy of c one cannot retain the Lorentz transformations for an infinitely small space-time region.

With this argument in mind one can readily see the dilemma that faced Abraham. If one retains the relativity principle it seems that one must choose the factor c to be a constant, and if c is further to be interpreted as the gravitational potential, then constant c corresponds to no gravitational field, and if one assumes that in small space-time regions one allows c to vary, thus permitting the existence of a gravitational field, then because of the above argument the Lorentz transformations would no longer be valid. Abraham appropriately rejected the validity of the principle of relativity as far as gravitational fields were concerned. Einstein, as we know, retained the principle of relativity, extending it through his principle of equivalence to include accelerations, which themselves were judged to be equivalent to gravitational fields. Historically one can therefore see the immense importance of the argument that went on between Abraham and Einstein (Einstein, 1912a, b)[2] for it pushed Einstein more and more to the point where he did not doubt the absolute validity of the path he was taking, and although he worked on some scalar theories of gravitation, no doubt spurred on by Abraham, who was sometimes extremely arrogant in his criticism, these only added to his conviction. Further evidence as to the unsuitability of Abraham's scalar theory was given by Einstein in a paper entitled "Zur Theorie des statischen Gravitationsfeldes" (Einstein, 1912c) in which he showed that given the important equations previously derived for the static gravitational field, namely,

$$c = c_0 + ax$$

where a is a constant previously explained

$$\Delta c = 0 \qquad \text{for material free space}$$

$$\Delta c = kc\rho_{m_0}$$

where k is a gravitational constant. The force density is given as before by the expression for f^g; for matter at rest we have, again as before

$$f^g = -\rho_{m_0}\operatorname{grad} c$$

then the following integral can be formed

$$\int f^g\,dt = -\int \rho_{m_0}\operatorname{grad} c\,dt = -\frac{1}{k}\int \frac{\Delta c}{c}\operatorname{grad} c\,dt$$

The integral is supposed taken about a space for which c is constant in the infinite and where consequently the law of action and reaction demands that it vanishes; otherwise matter which exists in the region under consideration will experience a self-imparted motion. The latter integral is in general not equal to zero and therefore the theory is not consistent, which led Einstein to doubt the whole foundations of the theory, and consequently to turn to some other form of gravitational theory. The venture that Einstein took into the gravitational scalar theories contains another interesting historical point in that it was in one of Einstein's papers of this period

[2] Both of these articles were Einstein's replies to remarks made by Max Abraham. Einstein finally stopped the communications because of Abraham's complete inability to listen to the reasoning behind the remarks and his personal animosity.

(Einstein, 1912c) that he first introduces the important equation

$$\delta\left\{\int\left(c^2\,dt^2-dx^2-dy^2-dz^2\right)^{1/2}\right\}=0$$

for the motion of a material point not acted upon by any external force in a static gravitational field. Abraham, on the other hand, pressed on with the theory arriving at the same equations as Einstein. The field equation obtained by Abraham was soon revised, in the light of Einstein's early arguments, to become[3]

$$c^{1/2}\Box c^{1/2}=2\rho_m c$$

which in the static case agreed with the equation obtained earlier by Einstein up to a factor of 2. Abraham realized that both sides of this equation had differing behavior under Lorentz transformation and consequently he gave up the validity of the principle of relativity and as a result moved away from the direction in which Einstein had been forced to move due to his belief in that principle (Abraham, 1912b).

While the debate between Einstein and Abraham was going on, other theories of gravitation were developing which perhaps might have given Einstein other roads to travel. We will give a short account of these and indicate why they, just like the theories of Abraham and the vector theories that preceded him, were equally unacceptable to Einstein. It seemed at that time, while the argument raged, that if the hypothesis that the quantity c, taken to represent the gravitational potential, was no longer required to be a universal constant led to such contradictory and unsatisfactory results, then the best thing to do was to adhere strictly to the concepts of special relativity and develop a theory of gravitation in which the velocity of light c is a constant and whose equations are invariant with respect to Lorentz transformations. The important theories of G. Nordström and G. Mie (Nordström 1912; 1913a,b; 1914; 1914/15) fall into this category, though the theory of the latter will not be dealt with because it evolved as part of a much wider theory that concerned itself with the foundations of matter and here is not the place to examine such a far-reaching theory.

Nordström derives, in reference to Abraham's early theory, the gravitational tensor $S_{ik}^{g(\phi)}$ through the scheme on p. 544 above from a scalar gravitational potential ϕ which satisfies the field equation $\rho_{m_0}=\Box\phi$, where again ρ_{m_0} denotes the rest mass density. Then it follows as before that the gravitational force density is

$$f_i^g=-\rho_{m_0}\frac{\partial\phi}{\partial x^i},\qquad i=1,2,3,4$$

[3]The ρ_m in this case of course takes into account the density of other fields as well as matter.

Now provided the body is small so that the gravitational field beyond its boundaries may be regarded as homogeneous, the gravitational force acting on a body of rest mass $\rho_{M_0} = \int \rho_{m_0}\, dV_0$, where the volume element is given by $dV = dV_0(1 - v^2/c^2)^{1/2}$, dV_0 being the rest volume, is

$$\mathbf{R}^g = \int \mathbf{F}^g\, dV = -\frac{\rho_{M_0}}{c}(c^2 - v^2)^{1/2} \operatorname{grad} \phi$$

A necessary conclusion of special relativity is that there can be no choice but to relate the scalar potential ϕ, according to equation (3), to ρ_{m_0}, which is an invariant with respect to Lorentz transformation, with the result that the gravitational force on a moving body is not proportional to its energy, but to its Lagrangian function

$$L = -\rho_{M_0} c (c^2 - v^2)^{1/2}$$

The importance in this choice of Lagrangian function lies in the fact that for small velocities it is not the sum of the potential and kinetic energy, but their difference that will determine the weight of the body. Now the scheme of things is such that ϕ can only enter into the Lagrangian function in the following way.

The momentum laws obtained through the method of Lagrangians must be formed such that

$$\frac{d\mathbf{S}}{dt} = \operatorname{grad} L$$

$$= \frac{\partial L}{\partial \phi} \operatorname{grad} \phi$$

i.e.,

$$\mathbf{R}^g = \frac{\partial L}{\partial \phi} \operatorname{grad} \phi = -\frac{d\rho_{M_0}}{d\phi} c (c^2 - v^2)^{1/2} \cdot \operatorname{grad} \phi$$

must agree with the former expression for $\mathbf{R}^g$, which results in the differential equation

$$\frac{d\rho_{M_0}}{d\phi} = \frac{\rho_{M_0}}{c^2}$$

whose integral is

$$\rho_{M_0} = \frac{M'}{c} e^{\phi/c^2}$$

where M' denotes a mass constant independent of ϕ.

In his first theory Nordström introduced the Lagrangian function

$$L = -M'e^{\phi/c^2}(c^2 - v^2)^{1/2}$$

from which the momentum and the energy are derived in the usual way:

$$\mathbf{S} = \frac{\partial L}{\partial v}\frac{\mathbf{m}}{v} = M'(c^2 - v^2)^{-1/2} \cdot e^{\phi/c^2} \cdot \mathbf{m}$$
$$\mathbf{E} = v\frac{\partial L}{\partial v} - L = M'c^2(c^2 - v^2)^{-1/2} \cdot e^{\phi/c^2} \qquad (5)$$

and the gravitational force becomes

$$\mathbf{R}^g = -\frac{M'}{c^2}e^{\phi/c^2} \cdot (c^2 - v^2)^{1/2} \cdot \operatorname{grad}\phi$$

and therefore the equation of motion of a material point or equivalent body, given by the equation $d\mathbf{s}/dt = \mathbf{R}^g$ becomes

$$\frac{d}{dt}\left\{e^{\phi/c^2} \cdot \frac{\mathbf{m}}{(c^2 - v^2)^{1/2}}\right\} = -\frac{1}{c^2}e^{\phi/c^2} \cdot (c^2 - v^2)^{1/2} \cdot \operatorname{grad}\phi$$

The situation is simplified if we revert to the static case where from equation (5) the energy integral

$$\frac{e^{\phi/c^2}}{(c^2 - v^2)^{1/2}} = \text{const}$$

is valid, so that in Nordström's theory the equations of motion of a material point in a static gravitational field are

$$\frac{d\mathbf{m}}{dt} = -\left(1 - \frac{v^2}{c^2}\right)\operatorname{grad}\phi$$

This compares with

$$\frac{d}{dt}\left(\frac{\mathbf{m}}{c^2}\right) = -\frac{1}{c}\operatorname{grad} c$$

obtained by Einstein and Abraham. The important and significant difference between them is illustrated when one considers the case $\mathbf{m} = C$. In

Nordström's theory the acceleration is zero, indicating that light travels in straight lines, while of course, in the latter theory it follows a curved path. This point together with the result that in Nordström's theory the gravitational mass is not proportional to the inertial mass, which as we have seen Einstein regarded as the foundation stone of any theory of gravitation, meant that this theory was not very well received, and Nordström consequently developed a theory on a similar basis except that the mass proportionality was achieved.

According to Einstein any gravitational theory must satisfy certain fundamental postulates. The requirements can be summarized as follows:

1. The conservation laws of momentum and energy must be fulfilled.
2. There must be equality of inertial and gravitational mass.
3. Any system of equations used must be covariant with respect to linear orthogonal substitutions (generalized Lorentz transformations).
4. The observed natural laws do not depend on the absolute values of the gravitational potential or potentials.

Without going into details Nordström's new theory satisfied all of these postulates except that in this theory the inertia of a body depended on bodies extraneous to the system under consideration, as Einstein would require, but this effect increased the further the body was removed from the others. This was, of course, an unacceptable result. However it is a historical fact that Einstein was deeply influenced by Nordström's theory, as can be seen in the development that took place in Einstein's own attempts in the following years.

4. EINSTEIN'S TENSOR THEORY

So far we have seen that if one takes either a scalar (in the four-dimensional sense) or a vector to represent the gravitational potential a theory results that is either inconsistent or does not meet the requirements listed above; so, for example, Abraham's theory does not meet the requirements of postulate 3. Einstein therefore took the next logical step, which was to choose a tensor to represent the gravitational potential. The essential difference between a tensor potential and a scalar potential lies in the fact that a theory founded on a tensor potential allows the existence of induced gravitational forces of the type first hypothesized by E. Mach and which Einstein convinced himself was a necessary component of any gravitational theory by once again turning to a direct analogy with electrodynamics. He was able to show that if a system is arranged as shown in Figure 1, the spherical layer, when given an acceleration, acts so as to produce a force which acts on the material point situated as its center. If the shell is at rest its presence increases the inertia of the mass at its center, and finally if the shell rotates then a coriolis force is formed inside the shell in such a way that a pendulum suspended inside is affected so that its plane of oscillation is carried along. Later Einstein was able to absorb this concept into his theory, which brought him great delight because he had at last quantified those thoughts of Mach on the subject which he had for many years himself thought to be deeply significant, from both a mathematical standpoint and a philosophical standpoint.

It was obvious to Einstein that the system of equations for the gravitational field, together with his deliberations concerning the extension of the principle of relativity, must be some generalization of the well-known Poisson equation

$$\Delta\phi = 4\pi k \rho_{m_0}$$

where ϕ is replaced by the ten quantities g_{ik} and ρ_{m_0} by a ten-component symmetrical tensor H^{ik} so as to form a system of the form

$$M^{ik} = \chi H^{ik}$$

where M^{ik} is formed from differential expressions of the g_{ik} and χ is a universal constant (gravitational constant).

One can at once see the similarity in form between Einstein's system of equations and those of the earlier vector and scalar theories, a point that we will take up later in more detail. The first set of equations that Einstein used

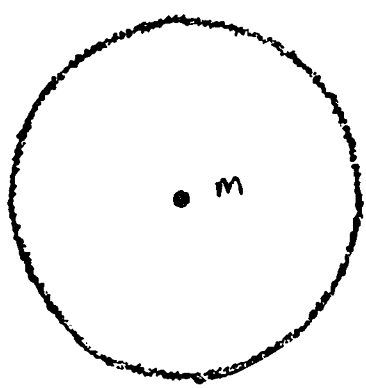


Fig. 1.

were as follows (cf. Einstein and Grossmann, 1913, p. 239):

$$\Delta^{ik}(g) = \chi(H^{ik} + D^{ik})$$

$$= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} \left\{ g^{ab} \sqrt{-g} \frac{\partial g^{ik}}{\partial x^b} \right\} - g^{ab} g_{cd} \frac{\partial g^{ic}}{\partial x^a} \frac{\partial g^{kd}}{\partial x^b} \tag{6}$$

and[4]

$$-2\chi D^{ik} = g^{ai} g^{bk} \frac{\partial g_{cd}}{\partial x^a} \frac{\partial g^{cd}}{\partial x^b} - \frac{1}{2} g^{ik} g^{ab} \frac{\partial g_{cd}}{\partial x^a} \frac{\partial g^{cd}}{\partial x^b} \tag{7}$$

The impulse-energy laws for material processes and the gravitational field together take the form

$$\frac{\partial}{\partial x^k} \left\{ (-g)^{1/2} g_{pi} (H^{ik} + D^{ik}) \right\} = 0$$

indicating conservation. H^{ik} and D^{ik} can be thought of as stress-energy tensors for the material processes and the gravitational field, respectively.

This system of equations achieved much of what was demanded by Einstein. For example, the relativity of inertia can be demonstrated in the first approximation by the following equations for the impulse I and the energy E:

$$I_x = \rho_{M_0} \left(1 + \frac{\chi}{8\pi} \int \frac{\rho_{m_0} dV}{r} \right) \frac{x}{c}, \qquad \text{etc.} \tag{8}$$

and

$$E = c\rho_{M_0} \left(1 - \frac{\chi}{8\pi} \int \rho_{m_0} \frac{dV}{r} \right) + \frac{1}{2} \frac{\rho_{M_0}}{c} \left(1 + \frac{\chi}{8\pi} \int \rho_{m_0} \frac{dV}{dr} \right) \tag{9}$$

The first part of the expression for E shows that the energy of a rest mass point decreases due to the presence of other masses in its vicinity and the same presence of other masses causes an increase in the inertia of the mass point, as required. There was, however, a fundamental problem associated with the scheme described above because in order that the laws of conserva-

[4]It is essential to realize that these equations are not yet tensor equations because they are valid only for linear transformations and not the continuous differentiable transformations.

tion of momentum (impulse) and energy be fulfilled one must demand that the equations be covariant with respect to certain "legitimate" transformations only, rather than with respect to arbitrary substitutions as was required if one was to accept the arguments that Einstein himself had put forward. The delight that Einstein obtained from his researches, and the frustrations, are clearly illustrated in an up till now unpublished letter from Einstein to Lorentz (Einstein, 1911; see Appendix). The date of this letter, August 14, 1911, shows just how advanced Einstein's thinking was at this stage as compared to others. The fact that his "house of cards," as he called his theory in a letter to Wladyslaw Natanson (Bergmann, 1968), rested on his principle of covariance and had appeared to have toppled caused him much disappointment and delayed his progress by about two years. Here, however, is not the place to discuss this side of the development of the theory; we return to consider how closely Einstein's theory resembled those preceding it.[5]

If we introduce the following quantities

$$T_i^k = (-g)^{1/2} g_{in} H^{nk}$$

and

$$t_i^k = (-g)^{1/2} g_{in} D^{nk}$$

then equation (6) takes the form

$$\frac{\partial}{\partial x^a}\left\{(-g)^{1/2} g^{ab} g_{ik} \frac{\partial g^{kn}}{\partial x^b}\right\} = \chi (T_i^n + t_i^n) \tag{10}$$

and equation (7) takes the form

$$-2\chi t_i^n = (-g)^{1/2}\left\{ g^{bn} \frac{\partial g_{cd}}{\partial x^i} \frac{\partial g^{cd}}{\partial x^b} - \delta_i^n g^{ab} \frac{\partial g_{cd}}{\partial x^a} \frac{\partial g^{cd}}{\partial x^b} \right\}$$

and the conservation laws take the form

$$\frac{\partial}{\partial x^a}\{T_i^a + t_i^a\} = 0$$

[5]It is interesting to read Einstein's reasons for only granting linear transformations, (see Einstein and Grossmann, 1913, p. 260) and to note that Grossmann realized that their field equations were not the logical ones to use, but rather ones including the Riemann–Christoffel tensor would be more appropriate. He, however, thought the latter were unsatisfactory in that, under certain conditions, reduction to the Poisson equation was not forthcoming.

This last equation has an identical form to that equation produced by Nordström to represent the conservation laws, and, despite its different foundations, is also remarkably similar in detail, though of course the functions occurring in Nordström's theory were based on a scalar potential and covariant with respect to generalized Lorentz transformations only. In fact there is a remarkable similarity in form between Einstein's theory and the scalar and vector theories that preceded it, as can be shown in the following way, in which generalizations are made so as to make the comparisons more visible.

Abraham had shown that if the equivalence of gravitational and inertial masses was to be fulfilled then one could not use a six-vector to describe the gravitational potential, and consequently, using the equations already developed in electrodynamics, he formed his system of equations based on a scalar potential. The field equations in Abraham's first theory can be written in the generalized form

$$P^i = -\frac{\partial\phi}{\partial x^i} \quad \text{and} \quad \frac{\partial Q^i}{\partial x^i} = -K\rho_{m_0}$$

where i runs from 1 to 4 and as before the imaginary coordinate system has been used. Another way of expressing things is to say that we have described the gravitational field by two vectors $(\mathbf{p}, ip^4)$ and $(\mathbf{Q}, iQ^4)$, then more explicitly we have

$$P^x = -\frac{\partial\phi}{\partial x}, \qquad P^y = -\frac{\partial\phi}{\partial y}, \qquad P^z = -\frac{\partial\phi}{\partial z}, \qquad P^4 = \frac{\partial\phi}{\partial t}, \quad \text{etc.}$$

The quantities p^i and Q^i are related to one another, for example, in a way similar to the relationship that exists between **D** and **E** in electrodynamics. In fact, in their respective theories Abraham and Nordström assumed that p^i and Q^i were identical to one another while Mie regarded them as distinct.

Now let us consider Einstein's theory in the light of this. We have

$$\frac{\partial}{\partial x^a}\left[(-g)^{1/2} g^{ab} g_{ik} \frac{\partial g^{kn}}{\partial x^b}\right] = \chi(T_i^n + t_i^n)$$

but

$$\frac{\partial g^{kn}}{\partial x^b} = -g^{kd} g^{ne} \frac{\partial g_{de}}{\partial x^b}$$

thus

$$\frac{\partial}{\partial x^a}\left((-g)^{1/2} g^{ab} g^{nc} \frac{\partial g_{ic}}{\partial x^b}\right) = -\chi(T_i^n + t_i^n)$$

Consider now the approximation that gave us equations (8) and (9), i.e., let

$$g_{ab} = \mathring{g}_{ab} + h_{ab}, \qquad g^{ab} = \mathring{g}^{ab} + h^{ab}$$

where the $\mathring{g}_{ab}$ have the constant values of special relativity. With this approximation equation (10) takes the form

$$\frac{\partial^2 h_{in}}{\partial x^{a^2}} = -\chi T_i^n$$

which is identical in form to the earlier equations of the previous theories if we write

$$Q^a = \frac{\partial h_{in}}{\partial x^a}$$

indicating that Einstein's theory differs from those theories simply in that ϕ has been replaced by a tensor potential and the mass density by a stress-energy tensor.

Indeed, even in the full theory this similarity is exhibited, though of course not so clearly, if one rewrites equation (10) to obtain

$$\frac{\partial}{\partial x^a}(\tau_i^{na}) = -\chi P_i^n$$

where

$$\tau_i^{na} = (-g)^{1/2} g^{ab} g^{nc} \frac{\partial g_{ic}}{\partial x^b}$$

$$P_i^n = T_i^n + t_i^n$$

5. CONCLUDING REMARKS

In conclusion therefore Einstein's theory shows the tremendous influence that electrodynamics had on the construction of his theory, even down to some detail. Even though he differed fundamentally from those who attempted to construct gravitational theories by direct analogy to electrodynamical theories, his own theory was ultimately not that different in form, but, however, completely different in its epistemology.

APPENDIX A: THE EINSTEIN LETTER

Zürich. 14. August

Hoch verehrter und lieber Herr Prof. Lorentz!

Ich danke Ihnen von Herzen für Ihre beiden Briefe, auch besonders für die freundliche Gratulation zu der neuen Stelle. Ich konnte der Versuchung nicht widerstehen, eine Stelle anzunehmen in der mir alle Verpflichtungen abgenommen sind, sodass ich mich ganz der Grüblerei hingeben kann.

Mit meiner Nachfolgerschaft ist es so: Wenn nicht alles trügt, wird Laue auf dieselbe Anspruch erheben, und es gibt gewiss niemanden, der mehr Anspruch darauf hätte. Aber ich glaube, dass Keesom für die Universität der richtige Mann wäre (an Laues Stelle), ja sogar, dass Keesom für die schweizerische Physik überhaupt ein wahrer Segen wäre, weil er die experimentelle

Technik der tiefen Temperaturen beherrscht. Daneben wird er gewiss auch als Theoretiker anregend wirken. Sie würden mich gewiss in dem Bestreben unterstützen, Keesom diesen Ruf zu verschaffen. Freilich ist mein Einfluss an der Universität nicht sehr gross. Aber ich habe schon an Laue über die Sache geschrieben und hoffe, auch seine Unterstützung zu finden. Ich würde mich <u>sehr</u> freuen, wenn Keesom eine~~r~~ seiner Tüchtigkeit entsprechende~~n~~ Anstellung finden würde. Sobald ich mich einigermassen in der Sache ~~orientieren~~ orientieren kann, schreibe ich Ihnen wieder. ~~in dieser Sache.~~

Mit der anderen Angelegenheit, Ihren Herrn Schwiegersohn betreffend, weiss ich im Augenblick nichts anzufangen; denn ich habe in Berlin weder ein Institut noch einen Assistenten.* Aber dass man <u>nun</u> einmal einen „Hans Dampf auf

* Ich bin nur Mitglied der Akademie und erhalte eine Bezahlung, die von der Akademie ~~stammt~~ ausgegeben wird. aber aus einer Schenkung eines Privatmannes stammt

allen Ga[redacted]en" aus mir gemacht hat, werde ich doch vielleicht einmal etwas für ihn thun können. Auch will ich Herrn Weiss davon sprechen, wenn er wiederkommt. Einstweilen freue ich mich darauf, Ihre lieben Kinder in Berlin näher kennen zu lernen und gratuliere Ihnen von Hagen als dem besten und zärtlichsten der Grossväter. Einstweilen bitte ich Sie, auch Ihnen allen meine freundschaftlichen Grüsse zu übermitteln. Sie können sicher sein, dass ich Ihre Anregung nicht aus dem Auge lassen werde.

Nun zur Gravitation. Ich bin beglückt darüber, dass Sie mit solcher Wärme sich unserer Untersuchung annehmen. Aber leider hat diese Sache doch noch so grosse Haken, dass mein Vertrauen in die Zulässigkeit der Theorie noch ein schwankendes ist. Befriedigend ist der Entwurf bis jetzt, soweit es sich um die Einwirkung des Gravitationsfeldes auf andere physikalische Vorgänge handelt. Denn der absolute Differential kalkül erlaubt hier die Aufstellung von Gleichungen, die beliebigen Substitutionen gegenüber kovariant sind. Das Gravitationsfeld ($g_{\mu\nu}$) erscheint sozusagen als das Gerippe an dem alles hängt. Aber die Gravitationsgleichungen selbst haben die Eigenschaft der allgemeinen Kovarianz leider nicht. Nur deren Kovarianz linearen Transformationen gegenüber ist gesichert. Nun beruht ~~ist~~ aber das ganze Vertrauen auf die ~~die ganze überzeugende Kraft der~~ Theorie auf der Überzeugung, dass Beschleunigung des Bezugssystems einem Schwerefeld äquivalent sei. Wenn also nicht alle Gleichungssysteme der Theorie, also auch Gleichungen (18) ausser den linearen noch andere Transformationen zulassen, so widerlegt die Theorie ihren eigenen Ausgangspunkt; sie steht dann in der Luft.

Bisher wollte es uns aber nicht gelingen, irgend welche nicht lineare Substitutionen anzugeben, denen gegenüber die Gleichungen

kovar

(18) kovariant wären. Zwei Möglichkeiten prinzipiell verschiedener Art kommen da in Betracht

1) Transformationen, welche von dem vorhandenen $g_{\mu\nu}$-Feld unabhängig sind, welche Ehrenfest als „selbständige Transformationen" bezeichnete; nur mit solchen hat sich meines Wissens bisher die Gruppentheorie beschäftigt.

2) Transformationen, deren p erst durch Differenzialgleichungen ~~aus~~ zu dem als gegeben zu betrachtenden $g_{\mu\nu}$-Feld zu bestimmen wären, die also dem vorhandenen $g_{\mu\nu}$-Feld angepasst werden müssen. Solche Transformationen sind – soviel ich weiss – noch nicht systematisch untersucht worden. („unselbständige Transformationen")

Die Existenz „selbständiger" nicht linearer Transformationen ist die einfachere Möglichkeit; diese scheint aber nicht zuzutreffen, ohne dass ich das zu beweisen wüsste. Es genügt aber schon die Existenz „unselbständiger" nicht linearer Transformationen, um mit der Aequivalenzhypothese nicht nachträglich in

Konflikt zu geraten.

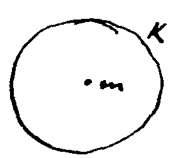
Prinzipiell liegt die Sache einfach. Man fragt: Welche Bedingungen müssen die p_{ik} einer Transformation erfüllen, damit

$$T_{\mu\nu} = \Delta_{\mu\nu}(\gamma) - \kappa\, \vartheta_{\mu\nu}$$

sich bei der Transformation wie ein Tensor transformiert? Man erhält so partielle Differenzialgleichungen für die p_{ik}. Es fragt sich, ob diese letzteren mit den Integrabilitätsbedingungen vereinbare Lösungen haben. – Will ich die Rechnung aber ausführen, so scheitere ich an der Kompliziertheit der Gleichungen. Sollte es sich zeigen lassen, dass nicht lineare Transformationen überhaupt nicht existieren, so verdiente die Theorie kein Vertrauen.

Sehr interessant ist es hingegen, dass die Gleichungen die Relativität der trägen Masse liefern. Es kommen nämlich folgende Dinge heraus:

1) Die Existenz einer trägen, ruhenden Kugelschale erhöht die Trägheit einer Masse m, die



sie umgibt
2) Eine Beschleunigung von K ~~liefert~~ induziert eine gleichsinnige beschleunigende Kraft, die auf m wirkt.
3) Rotiert K, so entsteht dadurch im Innern von K ein Coriolis-Feld, derart, dass ein im Innern von K angeordnetes Pendel so beeinflusst wird, dass seine Schwingungsebene mitgenommen wird.

Alle diese Effekte sind zwar wegen ihrer Kleinheit nicht der Prüfung zugänglich, sind aber an und für sich plausibel, wie Mach in seiner Mechanik so hübsch bei seiner Kritik von Newtons Prinzipien gezeigt hat.

Mit herzlichen Grüssen an Sie und Ihre werte Familie, auch von meiner Frau

Ihr ganz ergebener
A. Einstein.

APPENDIX B: THE TRANSLATION OF THE TECHNICAL PART OF THE EINSTEIN LETTER

Now to gravitation. I am made so happy that you are taking up its research with such warmth. But alas, the subject has still so many difficulties that my confidence in the admissibility of the theory is still uncertain. The model up till now is satisfactory, as long as it deals only with the influence of the gravitational field on other physical processes. Then the absolute differential calculus permits the formulation of equations which are covariant with respect to arbitrary substitutions. The gravitational field ($g_{\mu\nu}$) appears, so to speak, to be the framework on which all hangs. However, the gravitational equations themselves do not, alas, have the characteristic of general covariance.

Now confidence in the entire theory rests upon the conviction that acceleration of the reference system be equivalent to a gravitational field. If therefore all systems of equations of the theory and also therefore equation (18) do not admit, in addition to linear, still other transformations, then the theory, refuting its own origin, stands in the air. Till now it has been impossible for us to give any nonlinear substitution with respect to which equation (18) would be covariant. Two possibilities of a fundamentally different nature then come into consideration:

1. Transformations which are independent of the existing $g_{\mu\nu}$ field, which Ehrenfest described as "independent transformations," have, as far as I know, use only within group theory.
2. Transformations, whose p first fixed by differential equations of the $g_{\mu\nu}$ field regarded as given, which therefore must be adjusted to the existing $g_{\mu\nu}$ field. Such transformations—as far as I know—have still not yet been systematically investigated (dependent transformations).

The existence of "independent" nonlinear transformations is the simplest possibility; these however, without my being able to prove it, appear not to occur. It is enough, however, since only the existing dependent nonlinear transformations do not come into additional conflict with the equivalence hypothesis.

Fundamentally the situation is simple. One asks therefore which conditions must the p_{ik} of a transformation fulfil, in order that

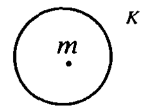
$$\Gamma_{\mu\nu} = \Delta_{\mu\nu}(\gamma) - K\theta_{\mu\nu}{}^{*}$$

itself transforms like a tensor under the transformation. One obtains then partial differential equations for the p_{ik}. It is a question of whether these latter have solutions compatible with the integrability conditions—I want to carry out the calculations but then I am frustrated by the complicatedness of the equations. It can be shown that if nonlinear transformations do not exist at all, then the theory merits no confidence.

*This equation and the equation (18) referred to in the letter are to be found in Einstein and Grossmann (1913), Chap. 4. p. 239.

On the contrary it is very interesting that the equations of relativity yield the inertial mass. The following thing is about to be published:

1. The existence of an inert resting spherical shell increases the inertia of a mass which is inside of it.
2. An acceleration of K induces a compatible accelerating force which acts on m.



3. If K rotates, then in this way a coriolis force is formed inside of K, in such a way that a pendulum suspended inside of K will be so influenced that its plane of oscillation will be taken along.

All these effects are indeed on account of their minuteness no proof but are in themselves plausible, as Mach has so elegantly shown through his critique of Newton's *Principia* in his *Mechanics*.

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